Numer. Math. 38, 447–453 (1982)

Numerische Mathematik

A Note on C° Galerkin Methods for Two-Point Boundary Problems

Miente Bakker

Mathematisch Centrum, Kruislaan 413, 1098 SJ Amsterdam

Summary. As is known [4], the C° Galerkin solution of a two-point boundary problem using piecewise polynomial functions, has $O(h^{2k})$ convergence at the knots, where k is the degree of the finite element space. Also, it can be proved [5] that at specific interior points, the Gauss-Legendre points the gradient has $O(h^{k+1})$ convergence, instead of $O(h^k)$. In this note, it is proved that on any segment there are $k-1$ interior points where the Galerkin solution is of $O(h^{k+2})$, one order better than the global order of convergence. These points are the Lobatto points.

Subject Classifications: AMS (MOS) 65 N 30; CR: 5.17.

1. Introduction

We consider the two-point boundary problem

$$\begin{aligned} &Lu \equiv -(p(x)\,u')' + q(x)\,u = f(x), \qquad x \in [0,1] = I; \\ &u(0) = u(1) = 0. \end{aligned} \tag{1}$$

We suppose that p, q and f are such that (1) has a unique and sufficiently smooth solution.

Let, for a constant integer N, $\Delta: 0 = x_0 < x_1 < \ldots < x_N = 1$ be a partition of I with

$$h = N^{-1}; \qquad x_j = jh; \qquad I_j = [x_{j-1}, x_j]$$

and let for a constant integer $k \geqq 2$ and for any interval $E \subset I$, $P_k(E)$ be the class of polynomials of degree at most k restricted to E.

We define for $m \geqq 0$ and $s \geqq 1$

0029-599X/82/0038/0447/$01.40

$$
\begin{aligned}
&W^{m,s}(E)=\{v \mid D^j v\in L^s(E), j=0,\dots,m\};\\
&H^m(E)=W^{m,2}(E);\\
&H_0^1(I)=\{v \mid v\in H^1(I); v(0)=v(1)=0\};\\
&M_0^k(\Delta)=\{v \mid v\in H_0^1(I); v\in P_k(I_j), j=1,\dots,N\};\\
&\|v\|_{W^{m,s}(E)}=\left[\sum_{j=0}^{m}\|D^j v\|_{L^s(E)}^2\right]^{\frac{1}{2}};\\
&\|v\|_{H^m(E)}=\left[\sum_{j=0}^{m}(D^j v, D^j v)_{L^2(E)}\right]^{\frac{1}{2}},
\end{aligned}
\tag{2}
$$

where D^j denotes d^j/dx^j. If $E=I$, we write (α,β) instead of $(\alpha,\beta)_{L^2(I)}$ and $\|\alpha\|_m$ instead of $\|\alpha\|_{H^m(I)}$.

Let $U\in M_0^k(\Delta)$ be the unique solution of

$$B(U,V)=(f,V), \quad V\in M_0^k(\Delta), \tag{3}$$

where $B\colon H_0^1(I)\times H_0^1(I)\to\mathbb{R}$ is defined by

$$B(u,v)=(pu',v')+(qu,v); \; u,v\in H_0^1(I). \tag{4}$$

We assume that B is strongly coercive, i.e. there exists a $C>0$ such that

$$B(v,v)\geqq C\|v\|_1^2, \qquad v\in H_0^1(I). \tag{5}$$

In the sequel, C, C_1, are generic positive constants not necessarily the same.

Lemma 1. *Let $u\in H_0^1(I)\cap H^{k+1}(I)$ be the solution of (1) and let $U\in M_0^k(\Delta)$ be the solution of (3). Then the error function $e(x)=u(x)-U(x)$ has the bounds*

$$
\begin{aligned}
&\|e\|_l\leqq Ch^{k+1-l}\|u\|_{k+1}, \qquad l=0,1;\\
&|e(x_j)|\leqq Ch^{2k}\|u\|_{k+1}, \qquad j=1,\dots,N-1;\\
&\|e\|_{L^\infty(I)}\leqq Ch^{k+1}\|u\|_{k+1}.
\end{aligned}
\tag{6}
$$

Proof. See [6], [4] and [7]. □

In the next §, we prove that the local order of convergence improves slightly at specific points interior to I_j, if u satisfies stricter smoothness requirements on the interior of I_j.

2. Order of Convergence at Lobatto Points

On the segment $[-1,+1]$, we define the Lobatto points $\sigma_0,\dots,\sigma_k$ by

$$(1-\sigma_l^2)\frac{d}{d\sigma}P_k(\sigma_l)=0, \qquad l=0,\dots,k, \tag{7}$$

where $P_k(\sigma)$ is the k-th degree Legendre polynomial. Associated to this polynomial is the quadrature formula (see [1, formula 25.4.32])

$$\int_{-1}^{+1} f(\sigma)\,d\sigma = \sum_{l=0}^{k} w_l f(\sigma_l) - \frac{(k+1)k^3 2^{2k+1}[(k-1)!]^4}{(2k+1)[(2k)!]^3} f^{(2k)}(s),\ s\in(-1,+1)$$
$$w_l = \frac{2}{k(k+1)[P_k(\sigma_l)]^2}, \qquad l=0,\dots,k. \tag{8}$$

From (7) and (8), we define

$$\xi_{jl} = x_{j-1} + \frac{h}{2}(1+\sigma_l); \qquad l=0,\dots,k;\ j=1,\dots,N;$$
$$(\alpha,\beta)_j^* = \frac{h}{2}\sum_{l=0}^{k} w_l\,\alpha(\xi_{jl})\,\beta(\xi_{jl}); \qquad \alpha,\beta\in W^{2k,\infty}(I_j); \qquad j=1,\dots,N; \tag{9}$$
$$(\alpha,\beta)_h = \sum_{j=1}^{N} (\alpha,\beta)_j^*.$$

We return to problems (1) and (3). It is known that

$$B(e,V)=0, \qquad V\in M_0^k(\Delta). \tag{10}$$

For any I_j, we define

$$M_0^k(I_j) = \{V \mid V\in M_0^k(\Delta), \operatorname{supp}(V)=I_j\}. \tag{11}$$

We temporarily drop the subscript j from the numbers ξ_{lj}. We define a natural basis $\{\phi_i\}_{i=1}^{k-1}$ for $M_0^k(I_j)$ by

$$\phi_i(\xi_l)=\delta_{il}, \qquad 1\leqq i,\ l\leqq k-1, \tag{12}$$

where δ_{il} is the Kronecker symbol. If we elaborate (10) for $V=\phi_i$, $i=1,\dots,k-1$, we get

$$(e, L\phi_i) = [p(x)\,e(x)\,\phi_i'(x)]_{\xi_0}^{\xi_k}, \qquad i=1,\dots,k-1. \tag{13}$$

Approximation of $(e, L\phi_i)$ by Lobatto quadrature yields

$$\begin{aligned}\sum_{l=1}^{k-1} w_l L\phi_i(\xi_l)\,e(\xi_l) &\\ = 2h^{-1}[p(x)\,e(x)\,\phi_i'(x)]_{\xi_0}^{\xi_k} &- w_0\,e(\xi_0)\,L\phi_i(\xi_0)\\ - w_k\,e(\xi_k)\,L\phi_i(\xi_k) &+ Ch^{2k}D^{2k}(eL\phi_i)(\xi\in I_j), \qquad i=1,\dots,k-1.\end{aligned} \tag{14}$$

This is a linear system for $e(\xi_1),\dots,e(\xi_{k-1})$. We have to prove the non-singularity of $(w_l L\phi_i(\xi_l))$ and to compute the order of the solution. We know that

$$\begin{aligned}hB(\phi_i,\phi_l) &= h(L\phi_i,\phi_l)\\ &= h^2\sum_{v=1}^{k-1} w_v L\phi_i(\xi_v)\,\phi_l(\xi_v) + Ch^{2k+2}D^{2k}(L\phi_i(\xi)\,\phi_l(\xi)),\ \xi\in I_j\\ &= h^2 w_l L\phi_i(\xi_l) + Ch^{2k+2}D^{2k}(L\phi_i(\xi)\,\phi_l(\xi)),\ \xi\in I_j.\end{aligned}$$

Hence we have

$$|hB(\phi_i, \phi_l) - h^2 w_l L\phi_i(\xi_l)| \leqq Ch^2. \tag{15}$$

This means that $M_1 = (h^2 w_l L\phi_i(\xi_l))$ is nearly equal to a symmetric positive definite matrix whose entries and positive eigenvalues are of $O(1)$ and consequently has an inverse with the same properties. If we represent $(hB(\phi_i, \phi_l))$ by M_2, we find that

$$M_1 = M_2 + h^2 M_3 = M_2(I + h^2 M_2^{-1} M_3).$$

where all M_i have entries of $O(1)$. Since the spectral radius of the perturbation matrix is of $O(h^2)$, it is evident by power series expansion that

$$M_1^{-1} = M_2^{-1} + h^2 M_4,$$

where the entries of M_4 are of $O(1)$. This proves that M_2^{-1} has entries of $O(1)$ and so we have that $(w_l L\phi_i(\xi_l))^{-1}$ has entries of $O(h^2)$.

We turn to the second part of our problem. The first three terms of the right hand side of (14) are of $O(h^{2k-2}\|u\|_{k+1})$. For the last term, we prove that

$$\|D^{2k}(eL\phi_i)\|_{L^\infty(I_j)} \leqq C\|e\|_{W^{2k,\infty}(I_j)} \|L\phi_i\|_{W^{2k,\infty}(I_j)}. \tag{16}$$

From [3], it can be proved that

$$\|D^l e\|_{L^\infty(I_j)} \leqq \begin{matrix} Ch^{k+1-l}\|u\|_{k+1}, & l \leqq k; \\ \|D^l u\|_{L^\infty(I_j)}, & l > k. \end{matrix} \tag{17}$$

Furthermore,

$$\|L\phi_i\|_{W^{2k,\infty}} \leqq Ch^{-k}, \tag{18}$$

hence we summarily have

$$\left|\sum_{l=1}^{k-1} w_l L\phi_i(\xi_l)\, e(\xi_l)\right| \leqq Ch^k[\|u\|_{k+1} h^{k-2} + \|u\|_{W^{2k,\infty}(I_j)}], \quad i = 1, \ldots, k-1. \tag{19}$$

This was the last step in the proof of

Theorem 1. *Let* $u \in H_0^1(I) \cap H^{k+1}(I) \cap \bigcap_{j=1}^{N} W^{2k,\infty}(I_j)$ *be the solution of* (1) *and let* $U \in M_0^k(\Delta)$ *be the solution of* (3). *Then the error function has the local error bound.*

$$|e(\xi_{jl})| \leqq Ch^{k+2}[\|u\|_{k+1} h^{k-2} + \|u\|_{W^{2k,\infty}(I_j)}], \quad j = 1, \ldots, N;\ l = 1, \ldots, k-1. \quad \square \tag{20}$$

3. Lobatto Quadrature

Usually, $B(\ ,\ )$ and $(\ ,\ )$ are to be evaluated by numerical quadrature. We will show that Lobatto quadrature leaves the order of convergence at the Lobatto points invariant.

We define

$$B_h(\alpha, \beta)=(p\alpha', \beta')_h+(q\alpha, \beta)_h; \qquad \alpha, \beta \in \bigcap_{j=1}^{N} W^{2k,\infty}(I_j), \tag{21}$$

where $(\ ,\ )_h$ is defined by (9).

Lemma 2. *Let $Y\in M_0^k(\Delta)$ be the solution of*

$$B_h(Y, V)=(f, V)_h, \qquad V\in M_0^k(\Delta) \tag{22}$$

and let $u\in H_0^1(I)\cap H^{k+1}(I)\cap \bigcap_{j=1}^{N} W^{2k,\infty}(I_j)$ be the solution of (1). *Then the error function $\eta=u-Y$ has the bounds*

$$|\eta(x_j)|\leqq Ch^{2k}\|f\|_{2k,\Delta}; \qquad j=1, \dots, N-1,$$

if h is small enough, with

$$\|f\|_{l,\Delta}=\left[\sum_{j=1}^{N}\|f\|^2_{H^l(I_j)}\right]^{\frac{1}{2}}. \tag{23}$$

Proof. See [4]. □

We now consider $\varepsilon(x)=U(x)-Y(x)$, where U is the solution of (3). From (3) and (22), we obtain for every I_j

$$\begin{aligned}|B(\varepsilon, V)|&\leqq|(f, V)-(f, V)_h|+|B_h(Y, V)-B(Y, V)|\\&\leqq Ch^{2k+1}\|V\|_{H^k(I_j)}[\|f\|_{H^{2k}(I_j)}+\|Y\|_{H^k(I_j)}], \qquad V\in M_0^k(I_j).\end{aligned}$$

If we take for V any of the basis functions ϕ_i of $M_0^k(I_j)$, as defined by (12), we have

$$|B(\varepsilon, \phi_i)|\leqq Ch^{k+1}[\|f\|_{H^{2k}(I_j)}+\|Y\|_{H^k(I_j)}], \qquad i=1, \dots, k-1. \tag{25}$$

Since

$$\begin{aligned}\sum_{l=1}^{k-1} w_l\, \varepsilon(\xi_l)\, L\phi_i(\xi_l)&=2h^{-1}B(\varepsilon, \phi_i)\\&\quad-w_0\, \varepsilon(\xi_0)\, L\phi_i(\xi_0)-w_k\, \varepsilon(\xi_k)\, L\phi_i(\xi_k)\\&\quad-\frac{2}{h}[p(x)\, \varepsilon(x)\, \phi_i'(x)]_{\xi_0}^{\xi_k}+Ch^{2k}D^{2k}(\varepsilon L\phi_i)\,(\xi\in I_j)\end{aligned} \tag{26}$$

and

$$\begin{aligned}\|D^{2k}(\varepsilon L\phi_i)\|_{L^\infty(I_j)}&\leqq C\|\varepsilon\|_{W^{k,\infty}(I_j)}\|\phi_i\|_{W^{k,\infty}(I_j)}\\&\leqq Ch^{-2k}\|\varepsilon\|_{L^\infty(I_j)}\leqq Ch^{-k+1}\|f\|_{2k,\Delta},\end{aligned} \tag{27}$$

we have

$$\left|\sum_{l=1}^{k-1} w_l\,\varepsilon(\xi_l)\,L\phi_i(\xi_l)\right| \leqq C_1\,h^k[\|f\|_{H^{2k}(I_j)}+\|Y\|_{H^k(I_j)}] + C_2\,h^{2k-2}\|f\|_{2k,\Delta}+C_3\,h^{k+1}\|f\|_{2k,\Delta}. \tag{28}$$

The nonsingularity of $(w_l\,L\phi_i(\xi_l))$ has already been proved, its inverse is of $O(h^2)$, hence we have

$$|\varepsilon(\xi_l)| \leqq C_1\,h^{k+2}[\|f\|_{H^{2k}(I_j)}+\|Y\|_{H^k(I_j)}]+C_2\,h^{k+3}\|f\|_{2k,\Delta}. \tag{29}$$

Since (see [3]).

$$\|Y\|_{H^k(I_j)} \leqq \|\eta\|_{H^k(I_j)}+\|u\|_{H^k(I_j)} \leqq Ch\|u\|_{k+1}+\|u\|_{H^k(I_j)} \leqq C\|u\|_{k+1}, \tag{30}$$

we can prove by combination of (20), (29) and (30)

Theorem 2. *Let* $u \in H_0^1(I) \cap H^{k+1}(I) \cap \bigcap_{j=1}^{N} W^{2k,\infty}(I_j)$ *be the solution of* (1) *and let* $Y \cap M_0^k(\Delta)$ *be the solution of* (22). *Then the error function* η *has the bounds*

$$|\eta(\xi_{lj})| \leqq C_1\,h^{k+2}[\|f\|_{H^{2k}(I_j)}+\|u\|_{k+1}]+C_2\,h^{k+3}\|f\|_{2k,\Delta}; \quad j=1,\ldots,N; \quad l=1,\ldots,k-1. \quad \square$$

4. Conclusions

We have found a weaker form of superconvergence at other points than the knots. The findings of this paper stress the important part that Lobatto points play in the C° Galerkin method for two-point boundary problems. This is especially true for $k=2$, since in that case the error is of $O(h^4)$ at all Lobatto points.

The results of this paper can be easily applied to the case of two-point initial boundary problems (see [2]) and probably to other cases, such as nonlinear boundary problems.

References

1. Abramowitz, A., Stegun, I.: Handbook of mathematical functions. Dover Publications, 1968
2. Bakker, M.: On the numerical solution of parabolic equations in a single space variable by the continuous time Galerkin method. SIAM J. Num. Anal. **17**, 161–177 (1980)
3. Ciarlet, P.G., Raviart, P.A.: General Lagrange and Hermite interpolation in R^N with applications to finite element methods. Arch. Rational. Mech. Anal. **46**, 177–199 (1972)
4. Douglas, J. Jr., Dupont, T.: Galerkin approximations for the two-point boundary problem using continuous, piecewise polynomial spaces. Num. Mat. **22**, 99–109 (1974)

5. Lesaint, P., Zlamal, M.: Superconvergence of the gradient of finite element solutions. R.A.I.R.O. **13**, 139–166 (1979)
6. Strang, G., Fix, G.J.: An analysis of the finite element method. Englewood Cliffs, New Jersey: Prentice-Hall 1973
7. Wheeler, M.F.: An optional L_∞ error estimate for Galerkin approximations to solutions of two-point boundary problems. SIAM J. Num. Anal. **10**, 914–917 (1973)

Received December 12, 1979; Revised April 12, 1981